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Translational friction coefficient of a permeable cylinder in a sheet of viscous fluid†

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Abstract. We calculate the translational friction coefficient and the translational diffusion coefficient of a permeable cylinder moving in a sheet of fluid which is embedded on both sides in a fluid of much lower viscosity. The result, which is an asymptotic expression valid in the limit of small ratios of the viscosities, is derived by the method of matched asymptotic expansions.

1. Introduction

In this paper we consider the following problem. A cylinder (radius a and height h) of constant permeability is in a state of uniform translation with respect to a viscous fluid (viscosity η). This fluid forms a sheet of thickness h in which the cylinder is constrained to move. The sheet is embedded on both sides in another fluid of a much lower viscosity (η'). We want to calculate, from the linearised Navier–Stokes equation, the translational friction coefficient f_T, which is defined as F/v_0, where F denotes the total force on the cylinder, and v_0 the relative velocity between cylinder and fluid.

This problem is relevant to the translational diffusion of a patch of cross-linked proteins in a cell membrane (Wiegel 1979b). For an impermeable cylinder in the limit $(a\eta'/h\eta)\to 0$, this problem has no solution because of the Stokes paradox. It follows from the theory of the present paper that this problem has no solution in the limit $(a\eta'/h\eta)\to 0$ for any finite value of the permeability either, i.e. even when the cylinder has a certain porosity the Stokes paradox still obtains. It was pointed out by Saffman and Delbrück (1975), for the case of an impermeable cylinder, that this problem does have a solution as long as $(a\eta'/h\eta)>0$. Saffman (1976) has presented the details of the calculation using a singular perturbation technique (cf Van Dyke 1975). In this paper we consider the case of a cylinder of finite length and with a finite permeability, and we calculate the leading term in f_T. For the sake of completeness, it should be pointed out that the rotational friction coefficient of a permeable cylinder in a viscous fluid has also been calculated. The result has been published elsewhere (Wiegel 1979a). The model considered in this paper is a generalisation of the model considered by Saffman (1976). Our model specifically enables one to study the effects of porosity on the translational diffusion coefficient of complexes in the cell membrane (cf Wiegel 1979b, c).

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An outline of the contents of this paper is as follows. In § 2 we develop the general theory and show that the problem can be reduced to a single ordinary integro-differential equation. In § 3 the asymptotic form of the solution for a radial distance $r \gg a$ is found. In § 4 the asymptotic form of the solution for $r \ll \eta h/\eta'$ is found. In § 5 the outer and inner asymptotic solutions are matched in the regime $a \ll r \ll \eta h/\eta'$, which leads to an explicit expression for the translational friction coefficient.

2. General theory

Consider a Cartesian system of coordinates (x, y, z) with the z axis along the axis of the cylinder. Shortly we shall also use cylindrical coordinates (r, ϕ, z). The sheet of fluid with viscosity η is located at $-h < z < 0$, while the other fluid with much lower viscosity η' is located at $z < -h$ and at $z > 0$.

First, we consider the pressure p and velocity $\boldsymbol{v} = (v_x, v_y, v_z)$ of the fluid in the half-space $z > 0$. Putting the cylinder at rest, we have to solve the linearised, time-independent Navier–Stokes equation

$$-\boldsymbol{\nabla} p + \eta' \Delta \boldsymbol{v} = 0 \tag{1}$$

and the incompressibility equation

$$\operatorname{div} \boldsymbol{v} = 0 \tag{2}$$

under the boundary condition that at large distances from the z axis $v_x \to -v_0$, $v_y \to 0$, $v_z \to 0$. For pressure and velocity we make the *ansatz*

$$v_x = -v_0 + (s(r, z) + t(r, z)) \cos^2 \phi - t(r, z), \tag{3}$$

$$v_y = (s(r, z) + t(r, z)) \cos \phi \sin \phi, \tag{4}$$

$$v_z = 0, \tag{5}$$

$$p = \text{constant}, \tag{6}$$

where s and t denote unknown functions with cylindrical symmetry which vanish for $r \to \infty$ or $z \to \infty$. Physically, $(\delta v)_r \equiv s(r, z) \cos \phi$ equals the component in the direction of increasing r values of the perturbation of the asymptotic velocity field, and $(\delta v)_\phi \equiv t(r, z) \sin \phi$ equals the component in the direction of increasing ϕ values. Substitution into the continuity equation (2) immediately leads to

$$t = -s - r\, \partial s/\partial r, \tag{7}$$

which relation reduces the number of unknown functions to one. Substituting the *ansatz* into (1) and eliminating t using the last relation, we find

$$\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) s(r, z) = 0. \tag{8}$$

The general solution, which is finite at $r = 0$ and vanishes if $r \to \infty$ or $z \to \infty$, is

$$s(r, z) = \int_0^\infty f(k) \frac{\mathrm{J}_1(kr)}{kr} \exp(-kz)\, \mathrm{d}k, \tag{9}$$

where the J_ν denote the Bessel functions of the first kind, and where f still has to be determined from the boundary condition at the interface at $z = 0$.

Second, we consider the fluid in the sheet $-h < z < 0$. In the biophysical application discussed elsewhere (Wiegel 1979b, c) this fluid consists of rod-like molecules (lipids) of length h, directed along the z axis. Hence the flow in the sheet is truly two-dimensional.

In the presence of a porous cylinder one has to use the (space–time) average pressure P and the average local velocity $\boldsymbol{V}$ of the fluid in the sheet. If the two fluids did not interact at their interfaces $z = 0$ and $z = -h$, the flow would follow from the Debye–Brinkman–Bueche equation

$$-\nabla P + \eta \Delta \boldsymbol{V} - (\eta/K)\boldsymbol{V} = 0, \tag{10}$$

where $K \equiv K(r)$ denotes the permeability. A microscopic derivation of this equation has been given by Felderhof and Deutch (1975a) and a macroscopic derivation by Wiegel and Mijnlieff (1976). Recent applications to the flow through polymer coils are due to Ooms *et al* (1970), Felderhof and Deutch (1975a, b), Felderhof (1975a, b), Wiegel and Mijnlieff (1977a, b) and Mijnlieff and Wiegel (1978).

In the presence of interaction between the two fluids at their interfaces, the fluid located at $z > 0$ exerts a force on the fluid in the sheet which has components (per unit area)

$$\sigma_x = \eta'(\partial v_x/\partial z)_{z=0}, \tag{11a}$$

$$\sigma_y = \eta'(\partial v_y/\partial z)_{z=0}, \tag{11b}$$

and an identical force derives from the second interface at $z = -h$. Hence P and $\boldsymbol{V}$ have to be determined from

$$-\nabla P + \eta \Delta \boldsymbol{V} - (\eta/K)\boldsymbol{V} + 2\boldsymbol{\sigma}/h = 0, \tag{12}$$

$$\operatorname{div} \boldsymbol{V} = 0. \tag{13}$$

For pressure and velocity in the sheet we now make the *ansatz*

$$V_x = -v_0 + (S(r) + T(r)) \cos^2 \phi - T(r), \tag{14}$$

$$V_y = (S(r) + T(r)) \cos \phi \sin \phi, \tag{15}$$

$$P = \eta \Pi(r) \cos \phi, \tag{16}$$

where S, T and Π denote unknown functions with cylindrical symmetry which vanish in the limit $r \to \infty$. Substitution into the continuity equation (13) leads to

$$T = -S - r\, \mathrm{d}S/\mathrm{d}r. \tag{17}$$

Substitution of the *ansatz* into (12) leads to two equations which, after some straightforward but tedious algebra, can be rewritten in the form

$$-\Pi' + S'' + \frac{3}{r}S' - \frac{S}{K} + \frac{v_0}{K} + \frac{2\eta'}{h\eta}\left(\frac{\partial s}{\partial z}\right)_{z=0} = 0, \tag{18}$$

$$-r\Pi'' - \Pi' + \Pi/r - r(K^{-1})'S + r(K^{-1})'v_0 = 0, \tag{19}$$

where the prime denotes differentiation with respect to r. These equations have to be solved under the boundary conditions: (i) $\Pi(0)$ and $S(0)$ should be finite; (ii) $\Pi(\infty) = S(\infty) = 0$; (iii) $s(r, 0) = S(r)$, which condition expresses the continuity of the flow field in the sheet with the two other flow fields at $z = 0$ and $z = -h$. The last term on the left-hand side of (18) is determined uniquely by the function $S(r)$ through (9) and boundary condition (iii).

In the rest of this paper we shall consider the special case of a homogeneous cylinder:

$$K(r)=\begin{cases}K_0 & (0<r<a), \qquad (20a)\\ \infty & (a<r). \qquad (20b)\end{cases}$$

The dimensionless parameter

$$\sigma \equiv a/\sqrt{K_0} \tag{21}$$

measures the ratio of the radius of the disc and the penetration length $\sqrt{K_0}$ over which fluid flow effectively penetrates the porous material (this quantity should of course not be confused with the shear force $\boldsymbol{\sigma}$ in (11) and (12)). With a discontinuity in $K(r)$ at $r=a$, equation (12) shows that the pressure, velocity and all first derivatives of the velocity should be continuous at the surface of the disc, but that the derivative of the pressure and/or the second derivatives of the velocity may be discontinuous. Equation (19) now reads

$$\Pi''+\Pi'/r-\Pi/r^2=0, \tag{22}$$

with the solution

$$\Pi(r)=\begin{cases}Lr & (0<r<a), \qquad (23a)\\ La^2/r & (a<r), \qquad (23b)\end{cases}$$

where L is a constant. Equation (18) then reads

$$-L+S''+\frac{3}{r}S'-\frac{S}{K_0}+\frac{v_0}{K_0}-\frac{2\eta'}{h\eta r}\int_0^\infty f(k)\mathrm{J}_1(kr)\,\mathrm{d}k=0 \qquad (0<r<a), \tag{24}$$

$$\frac{La^2}{r^2}+S''+\frac{3}{r}S'-\frac{2\eta'}{h\eta r}\int_0^\infty f(k)\mathrm{J}_1(kr)\,\mathrm{d}k=0 \qquad (a<r), \tag{25}$$

$$S(r)=\int_0^\infty f(k)\frac{\mathrm{J}_1(kr)}{kr}\,\mathrm{d}k. \tag{26}$$

It is easy to verify that these equations have no solution which satisfies the boundary conditions at $r=0$ and $r=\infty$ in the limit $(a\eta'/h\eta)\to 0$. Hence the Stokes paradox also holds for a porous cylinder of infinite length. For $0<(a\eta'/h\eta)\ll 1$, however, the equations do have a solution which can be found using singular perturbation theory.

3. The outer asymptotic expansion

For $r\gg a$, the force density $-(\eta/K)\boldsymbol{V}$ in the Debye–Brinkman–Bueche equation (12) can be replaced by a sharply peaked force density $\boldsymbol{F}$ in the origin, with components

$$F_x=(F/\pi hr)\delta(r), \tag{27a}$$

$$F_y=0. \tag{27b}$$

Here F denotes the magnitude of the total force which the cylinder exerts on the fluid, and we have made use of the fact that the delta function $\delta(r)$ is an even function. The asymptotic behaviour of the flow field for $r\gg a$ can thus be found from the equation

$$-\boldsymbol{\nabla}P+\eta\Delta\boldsymbol{V}+\boldsymbol{F}+2\boldsymbol{\sigma}/h=0. \tag{28}$$

It is straightforward, but somewhat tedious, to solve this equation by the methods of the previous section. Instead of (18) and (19) we now have the equations

$$-\Pi'+S''+\frac{3}{r}S'+\frac{F}{\pi\eta hr}\delta(r)+\frac{2\eta'}{h\eta}\left(\frac{\partial s}{\partial z}\right)_{z=0}=0, \tag{29}$$

$$-r\Pi''-\Pi'+\frac{\Pi}{r}+\frac{F}{\pi\eta h}r\left(\frac{1}{r}\delta(r)\right)'=0. \tag{30}$$

The last equation can be integrated immediately; using the boundary condition on Π at infinity one finds

$$-\Pi'+(F/\pi\eta hr)\delta(r)=\Pi/r. \tag{31}$$

The solution of the equation which has the appropriate behaviour near $r=0$ is

$$\Pi(r)\simeq F/2\pi\eta hr \qquad (r\gg a). \tag{32}$$

Note that this asymptotic behaviour of the pressure has the same form as the exact result (23*b*).

Upon substitution of the last two equations, (29) takes the form

$$S''+\frac{3}{r}S'+\frac{F}{2\pi\eta hr^2}+\frac{2\eta'}{h\eta}\left(\frac{\partial s}{\partial z}\right)_{z=0}=0. \tag{33}$$

Taking the Hankel transform and using (9) and (26) we find the solution in the form of a definite integral:

$$S(r)\simeq\frac{F}{2\pi\eta h}\int_0^\infty\left(\xi^2+\frac{2\eta' r\xi}{h\eta}\right)^{-1}\mathrm{J}_1(\xi)\,\mathrm{d}\xi \qquad (r\gg a). \tag{34}$$

The solution also obeys the proper boundary condition for $r\to\infty$. Using the recurrence relations for the Bessel functions (Abramowitz and Stegun 1968, equation 9.1.27) the integral can be written as the sum of two terms:

$$\int_0^\infty\left(\xi^2+\frac{2\eta' r\xi}{h\eta}\right)^{-1}\mathrm{J}_1(\xi)\,\mathrm{d}\xi$$
$$=\frac{1}{2}\int_0^\infty\left(\xi+\frac{2\eta' r}{h\eta}\right)^{-1}\mathrm{J}_0(\xi)\,\mathrm{d}\xi+\frac{1}{2}\int_0^\infty\left(\xi+\frac{2\eta' r}{h\eta}\right)^{-1}\mathrm{J}_2(\xi)\,\mathrm{d}\xi. \tag{35}$$

The second integral on the right-hand side converges even for $r=0$, and has the value (Gradshteyn and Ryzhik 1965, equation 6.561.17)

$$\frac{1}{2}\int_0^\infty\left(\xi+\frac{2\eta' r}{h\eta}\right)^{-1}\mathrm{J}_2(\xi)\,\mathrm{d}\xi\simeq\frac{1}{4} \qquad \left(r\ll\frac{h\eta}{\eta'}\right). \tag{36}$$

The first integral on the right-hand side has the value (Gradshteyn and Ryzhik 1965, equation 6.562.2)

$$\frac{1}{2}\int_0^\infty\left(\xi+\frac{2\eta' r}{h\eta}\right)^{-1}\mathrm{J}_0(\xi)\,\mathrm{d}\xi=\tfrac{1}{4}\pi\left(\mathrm{H}_0\left(\frac{2\eta' r}{h\eta}\right)-\mathrm{N}_0\left(\frac{2\eta' r}{h\eta}\right)\right), \tag{37}$$

where the N_ν denote the Neumann functions and the H_ν the Struve functions. Using the asymptotic expansions of these functions for small arguments, as given by Gradshteyn and Ryzhik (1965, equations 8.550 and 8.403.2), and combining the last four equations, we find for the inner limit of the outer asymptotic expansion of the flow

$$S(r) \simeq (F/4\pi\eta h)[\ln(h\eta/\eta' r) - \gamma + \tfrac{1}{2}] \qquad (a \ll r \ll h\eta/\eta'), \tag{38}$$

where $\gamma = 0{\cdot}5772$ denotes Euler's constant. The constant F will be determined shortly.

4. The inner asymptotic expansion

For $r \ll h\eta/\eta'$, the last term on the left-hand side of (24) and (25) can be neglected with respect to the terms $S'' + (3/r)S'$. Hence the inner asymptotic expansion can be obtained from the equations

$$-L + S'' + (3/r)S' - S/K_0 + v_0/K_0 = 0 \qquad (0 < r < a), \tag{39}$$

$$La^2/r^2 + S'' + (3/r)S' = 0 \qquad (a < r). \tag{40}$$

The solution of these equations which is finite for $r \downarrow 0$ (the inner asymptotic expansion) is

$$S(r) \simeq v_0 - LK_0 + A\, \mathrm{I}_1(r/\sqrt{K_0})/(r/\sqrt{K_0}) \qquad (0 < r < a), \tag{41}$$

$$S(r) \simeq \alpha/r^2 + \beta - \tfrac{1}{2}La^2 \ln r \qquad (a < r), \tag{42}$$

where A, α and β are constants and where the I_ν denote the modified Bessel functions.

As has been noted already in the Introduction, the pressure, velocity and all first derivatives of the velocity should be continuous at $r = a$. With (14), (15) and (17) this implies that S, S' and S'' should be continuous at $r = a$. With the explicit results (41) and (42) this leads to three equations from which the three constants α, β and A can be determined as functions of L. This results in

$$A = -La^2/\sigma \mathrm{I}_1(\sigma), \tag{43}$$

$$2\alpha/a^2 = -La^2(\tfrac{1}{2} + 2/\sigma^2 - \mathrm{I}_0(\sigma)/\sigma\mathrm{I}_1(\sigma)), \tag{44}$$

$$\beta = v_0 + La^2(-1/\sigma^2 + \tfrac{1}{2}\ln a + \tfrac{1}{4} - \mathrm{I}_0(\sigma)/2\sigma\mathrm{I}_1(\sigma)). \tag{45}$$

The outer limit of the inner asymptotic expansion is found from (42):

$$S(r) \simeq \beta - \tfrac{1}{2}La^2 \ln r \qquad (a \ll r \ll h\eta/\eta'). \tag{46}$$

The unknown constant L is determined in the next section.

5. Asymptotic matching and concluding remarks

The method of matched asymptotic expansions has been discussed in full detail by Van Dyke (1975). The method relies essentially on the observation that, for

$$\eta h/\eta' a \gg 1, \tag{47}$$

a regime of r values ($a \ll r \ll h\eta/\eta'$) exists in which both the outer asymptotic expansion (34) and the inner asymptotic expansion (41), (42) are good approximations. Asymptotic matching is accomplished using the rule

$$\text{(inner limit of the outer expansion)} = \text{(outer limit of the inner expansion)}. \tag{48}$$

Substituting (38) and (46), one finds two equations in the two unknown quantities L and F, which can be solved to give

$$F=4\pi\eta h v_0[-\gamma+\ln(\eta h/\eta' a)+2/\sigma^2+\mathrm{I}_0(\sigma)/\sigma\mathrm{I}_1(\sigma)]^{-1}, \tag{49}$$

$$La^2=2v_0[-\gamma+\ln(\eta h/\eta' a)+2/\sigma^2+\mathrm{I}_0(\sigma)/\sigma\mathrm{I}_1(\sigma)]^{-1}. \tag{50}$$

Hence, if (47) holds, the translational friction coefficient is given by

$$f_\mathrm{T}\simeq 4\pi\eta h[-\gamma+\ln(\eta h/\eta' a)+2/\sigma^2+\mathrm{I}_0(\sigma)/\sigma\mathrm{I}_1(\sigma)]^{-1}. \tag{51}$$

Using the Einstein relation (52), this gives for the translational diffusion coefficient

$$D_\mathrm{T}=k_\mathrm{B}T/f_\mathrm{T} \tag{52}$$

$$\simeq (k_\mathrm{B}T/4\pi\eta h)[-\gamma+\ln(\eta h/\eta' a)+2/\sigma^2+\mathrm{I}_0(\sigma)/\sigma\mathrm{I}_1(\sigma)], \tag{53}$$

where k_B denotes Boltzmann's constant and T the absolute temperature.

In the limit $\sigma\to\infty$ the cylindrical disc becomes impermeable, and we find

$$D_\mathrm{T}(\infty)\simeq(k_\mathrm{B}T/4\pi\eta h)[-\gamma+\ln(\eta h/\eta' a)]. \tag{54}$$

This is the result of Saffman and Delbrück (1975) and Saffman (1976) for the diffusion coefficient of a hard disc. Our method, therefore, generalises Saffman's method to permeable objects. If, on the other hand, $\sigma\ll 1$, our result simplifies to

$$D_\mathrm{T}\simeq k_\mathrm{B}T/\pi\eta h\sigma^2 \qquad (\sigma\ll 1). \tag{55}$$

This expression can also be found using ordinary perturbation theory.

Equations (51) and (53) form the main results of this paper. They can be used to calculate translational diffusion coefficients of permeable complexes of cross-linked immunoglobulins in the cell membrane—a calculation which was not possible up till now. These more biophysical applications of our results form the subject of two separate publications (Wiegel 1979b, c).

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